

The probability of unusually large components in the near-critical Erdős-Rényi graph

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Abstract

We give detailed asymptotics for the probability that there is an unusually large component in a critical Erdős-Rényi graph, i.e. of size $an^{2/3}$ for large a . Our results hold throughout the critical window and somewhat beyond, and are an extension of existing results due to Pittel. We also provide asymptotics for the distribution of the size of the component containing a particular vertex.

1 Introduction

Much is known about the component sizes of the Erdős-Rényi graph within the critical window, $G(n, p)$ with $p = \frac{1}{n}(1 + \lambda n^{-1/3})$. Some of the most precise estimates are due to Pittel [18]. In particular we are interested in the asymptotic probability that the largest component is unusually large, i.e. $\mathbb{P}(L_1 \geq an^{2/3})$ where L_1 is the size (number of vertices) of the largest component and $a \rightarrow \infty$. In [18, Proposition 2], Pittel gave an asymptotic for $\lim_{n \rightarrow \infty} \mathbb{P}(L_1 \geq an^{2/3})$ as $a \rightarrow \infty$ for fixed λ .

In fact the method that Pittel used to obtain his result is substantially more robust than the result itself implies; the asymptotic holds for a wider range of values of λ and a , which may for example go to infinity with n . The same method can also be used to approximate $\mathbb{P}(L_1 = k)$ for $k \approx an^{2/3}$ with a large. The aim of this article is to make these more general results precise, and rehash the methods of Pittel (which in turn owe much to another paper of Łuczak, Pittel and Wierman [16]) to prove them. Our proofs consist largely of elementary approximations applied very precisely; but we have found the results useful, in particular for work on a dynamical version of the Erdős-Rényi graph [20], and felt that they should be stated clearly in a general form.

We should remark that Pittel proved several other results in his article [18], which we make no attempt to rework. Our results are based around his Proposition 2. We also note here that the constant $(2\pi)^{1/2}$ in the denominator of Pittel's result should in fact be $(9\pi/8)^{1/2}$; there is a small oversight between the last line of page 266 and the first line of page 267 in [18].

We mention also a very nice paper by Nachmias and Peres [17] which gives bounds on $\mathbb{P}(L_1 \geq an^{2/3})$ via martingale arguments, valid—at least when $\lambda = 0$ —for any $n > 1000$ and $a > 8$. Theoretically it should also be possible to extract concrete bounds (that is, specific error bounds and a value of N such that the bounds hold for all $n \geq N$) from our proofs, but we felt it best not to include these in order to keep the article to a reasonable length.

Let $p_{n,\lambda} = n^{-1} + \lambda n^{-4/3}$, and let $\mathbb{P}_{n,\lambda}$ be the law of an Erdős-Rényi random graph with n vertices and edge probability $p_{n,\lambda}$. (We allow λ to depend on n .) To state our main results, define

$$G_\lambda(x) = \frac{x^3}{8} - \frac{\lambda x^2}{2} + \frac{\lambda^2 x}{2} = \frac{x}{8}(x - 2\lambda)^2,$$

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$$\mathcal{E}_1 = \mathcal{E}_1(k, n, \lambda) = \frac{k^4}{n^3} + \frac{|\lambda|}{n^{1/12}} + \frac{|\lambda|^3 k}{n} + \frac{n^{2/3}}{k} + \frac{1}{n^{1/10}}$$

and

$$\mathcal{E}_2 = \mathcal{E}_2(n) = n^{1/4} e^{-n^{1/4}/80}.$$

Theorem 1. *Suppose that $n \geq 81$, $-n^{1/12} \leq \lambda \leq \frac{1}{5}n^{1/12}$, and $(3\lambda \wedge 1)n^{2/3} \leq k \leq n^{3/4}$. Let $a = k/n^{2/3}$. Then*

$$(a) \quad \mathbb{P}_{n,\lambda}(L_1 = k) = \frac{k^{1/2}}{(8\pi)^{1/2}n} e^{-G_\lambda(a)} (1 + O(\mathcal{E}_1)) + O(\mathcal{E}_2),$$

$$(b) \quad \mathbb{P}_{n,\lambda}(L_1 \geq k) = \frac{a^{1/2}}{(8\pi)^{1/2}G'_\lambda(a)} e^{-G_\lambda(a)} (1 + O(\mathcal{E}_1)) + O(\mathcal{E}_2).$$

Notice in particular that in (b), if λ is fixed and $a \rightarrow \infty$ then $G'_\lambda(a) = 3a^2/8 + O(a)$, so the correct constant in the denominator of [18, Proposition 2] should be $(9\pi/8)^{1/2}$ as mentioned above.

We also give a related result that provides information on the size of the component containing a particular vertex v . Let $\mathcal{C}(v)$ be the connected component containing v , and write $|\mathcal{C}(v)|$ for the number of vertices in $\mathcal{C}(v)$.

Theorem 2. *Suppose that $n \geq 81$, $|\lambda| \leq n^{1/12}$, and $n^{2/3} \leq k \leq n^{3/4}$. Let $a = k/n^{2/3}$. Then*

$$(a) \quad \mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| = k) = \frac{k^{3/2}}{(8\pi)^{1/2}n^2} e^{-G_\lambda(a)} (1 + O(\mathcal{E}_1)).$$

Suppose in addition that $\lambda \leq \frac{1}{5}n^{1/12}$ and $(3\lambda \wedge 1)n^{2/3} \leq k$ (as in Theorem 1). Then

$$(b) \quad \mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| \geq k) = \frac{a^{3/2}}{(8\pi)^{1/2}n^{1/3}G'_\lambda(a)} e^{-G_\lambda(a)} (1 + O(\mathcal{E}_1)) + O(\mathcal{E}_2).$$

Some further results are given in Section 8 which may be useful: Proposition 22 provides an asymptotic for the probability that $\mathcal{C}(v)$ has k vertices and $k+l$ edges, and Proposition 23 gives slightly rougher bounds on $\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| = k)$ when $k \leq n^{2/3}$.

Let $X(k, k+l)$ be the number of components with k vertices and $k+l$ edges. Our tactic will be to estimate $\mathbb{E}_{n,\lambda}[X(k, k+l)]$ very precisely and then show that $\mathbb{E}_{n,\lambda}[X(k, k+l)^2] \approx \mathbb{E}_{n,\lambda}[X(k, k+l)]$. We will often use the expansion

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (1)$$

to bound $(1+x)^y = e^{y \log(1+x)}$, including as many terms in the expansion as are required to give an accurate estimate. We will also regularly apply Stirling's formula [19],

$$j! = (2\pi)^{1/2} j^{j+1/2} e^{-j} (1 + O(1/j)).$$

There is a rich literature on Erdős-Rényi random graphs and related models, beginning with the work of Erdős and Rényi themselves [7, 8, 9]. A good introduction to the general area is provided by the three books [5, 6, 14]. More directly relevant to this work, besides the articles by Łuczak, Pittel and Wierman [16] and Pittel [18], is a paper by van der Hofstad, Kager and Müller [11] which gives a local limit theorem for the size of the k largest components for arbitrary k .

As mentioned above, we use combinatorial methods to estimate the distribution of component sizes. Another approach could be to exploit the link between Erdős-Rényi random graphs and excursions of Brownian motion with parabolic drift, first identified by Aldous [3]. Aldous used a breadth-first walk to explore the graph, and showed that the sizes of the largest components, when rescaled by $n^{2/3}$, converge in an appropriate sense to some limit, which he described in detail. This link has since been built upon in various ways. For example, a sharpening of Pittel's result was obtained by van der Hofstad, Janssen and van Leeuwen [10], who also used the same

tools to investigate critical SIR epidemics. Addario-Berry, Broutin and Goldschmidt [2] showed that in fact the components themselves (rather than just their sizes) converge, when rescaled, to metric spaces characterized by excursions of Brownian motion with parabolic drift decorated by a Poisson point process; they then used this relationship to give various distributional properties of the components [1].

2 Approximating the expected value of $X(k, k+l)$

Define

$$F_\lambda(x) = x^3/6 - \lambda x^2/2 + \lambda^2 x/2$$

and let $C(k, k+l)$ be the number of connected graphs on k vertices with $k+l$ edges. Our main result in this section is the following.

Proposition 3. *Suppose that $n \geq 16$, $|\lambda| \leq n^{1/12}$, $1 \leq k \leq n^{3/4}$ and $l \leq 4n^{1/4}$. Then*

$$\mathbb{E}_{n,\lambda}[X(k, k+l)] = \frac{C(k, k+l)}{(2\pi)^{1/2} n^l k^{k+1/2}} e^{-F_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_3))$$

where

$$\mathcal{E}_3 = \mathcal{E}_3(k, n, l, \lambda) = \frac{k^4}{n^3} + \frac{|\lambda|}{n^{1/3}} + \frac{k|\lambda|^3}{n} + \frac{k}{n} + \frac{1}{k}.$$

Furthermore, there exists a finite constant c such that for any $n \geq 16$, $|\lambda| \leq n^{1/12}$, $1 \leq k \leq n-1$ and $l \geq -1$,

$$\mathbb{E}_{n,\lambda}[X(k, k+l)] \leq \frac{c}{k} \left(\frac{k^3}{n^2(l \vee 1)} \right)^{l/2} e^{-F_\lambda(k/n^{2/3}) + \frac{1}{3}\lambda^3 k n^{-1}} \left(\frac{1 + \lambda n^{-1/3}}{1 - 2n^{-1}} \right)^l.$$

To prove this, we start with the simple observation

$$\mathbb{E}_{n,\lambda}[X(k, k+l)] = \binom{n}{k} C(k, k+l) p_{n,\lambda}^{k+l} (1 - p_{n,\lambda})^{\binom{k}{2} - (k+l) + k(n-k)}. \quad (2)$$

In a series of simple lemmas, we approximate the terms $\binom{n}{k}$, $p_{n,\lambda}^{k+l}$ and $(1 - p_{n,\lambda})^{\binom{k}{2} - (k+l) + k(n-k)}$.

Lemma 4. *If $n \geq 16$ and $k^4 \leq n^3$, we have*

$$\binom{n}{k} = \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \exp\left(k - \frac{k^2}{2n} - \frac{k^3}{6n^2}\right) \left(1 + O\left(\frac{1}{k} + \frac{k}{n} + \frac{k^4}{n^3}\right)\right).$$

Furthermore, there exists a finite constant c such that for any $k = 1, \dots, n-1$,

$$\binom{n}{k} \leq \frac{cn^k}{k^{k+1/2}} \exp\left(k - \frac{k^2}{2n} - \frac{k^3}{6n^2}\right).$$

Proof. By Stirling's formula,

$$\begin{aligned} \binom{n}{k} &= \frac{n^{n+1/2}}{(2\pi)^{1/2} k^{k+1/2} (n-k)^{n-k+1/2}} (1 + O(\frac{1}{k} + \frac{1}{n-k})) \\ &= \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \left(1 - \frac{k}{n}\right)^{k-n-1/2} (1 + O(\frac{1}{k} + \frac{1}{n-k})). \end{aligned}$$

In the case when $k^4 \leq n^3$ (note that since $n \geq 16$ we also have $k \leq n/2$), we use the expansion (1) to get

$$\binom{n}{k} = \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \left(\exp\left(-\frac{k}{n} - \frac{k^2}{2n^2} - \frac{k^3}{3n^3} + O\left(\frac{k^4}{n^4}\right)\right) \right)^{k-n-1/2} (1 + O(\frac{1}{k})).$$

Simplifying,

$$\begin{aligned} \binom{n}{k} &= \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \exp\left(-\frac{k^2}{n} - \frac{k^3}{2n^2} + k + \frac{k^2}{2n} + \frac{k^3}{3n^2} + O\left(\frac{k^4}{n^3}\right) + O\left(\frac{k}{n}\right)\right) (1 + O(\frac{1}{k})) \\ &= \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \exp\left(k - \frac{k^2}{2n} - \frac{k^3}{6n^2}\right) \left(1 + O\left(\frac{1}{k} + \frac{k^4}{n^3} + \frac{k}{n}\right)\right). \end{aligned}$$

This gives the first part of the lemma. For the second part, we note that when we used (1), if we had written out the expansion in full we would have obtained

$$\begin{aligned} \binom{n}{k} &= \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \left(\exp\left(-\sum_{j=1}^{\infty} \frac{k^j}{jn^j}\right)\right)^{k-n-1/2} (1 + O(\frac{1}{k} + \frac{1}{n-k})) \\ &= \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \exp\left(k - \sum_{j=1}^{\infty} \frac{k^{j+1}}{j(j+1)n^j}\right) (1 + O(\frac{1}{k} + \frac{1}{n-k} + \frac{k}{n})) \\ &\leq \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \exp\left(k - \frac{k^2}{2n} - \frac{k^3}{6n^2}\right) (1 + O(\frac{1}{k} + \frac{1}{n-k} + \frac{k}{n})). \end{aligned}$$

Since $1 \leq k \leq n-1$, the $O(\cdot)$ terms contribute at most a constant factor. □

Lemma 5. Suppose that $|\lambda| \leq n^{1/3}/2$, $l|\lambda| \leq 4n^{1/3}$ and $k|\lambda|^3 \leq 3n$. Then

$$p_{n,\lambda}^{k+l} = \frac{1}{n^{k+l}} e^{\lambda kn^{-1/3} - \frac{1}{2}\lambda^2 kn^{-2/3}} \left(1 + O\left(\frac{\lambda l}{n^{1/3}}\right) + O\left(\frac{k\lambda^3}{n}\right)\right).$$

Furthermore, for any λ , l , k and n ,

$$p_{n,\lambda}^{k+l} = \frac{1}{n^{k+l}} e^{\lambda kn^{-1/3} - \frac{1}{2}\lambda^2 kn^{-2/3} + \frac{1}{3}\lambda^3 kn^{-1}} (1 + \lambda n^{-1/3})^l.$$

Proof. First we write

$$p_{n,\lambda}^{k+l} = \frac{1}{n^{k+l}} (1 + \lambda n^{-1/3})^{k+l} = \frac{1}{n^{k+l}} (1 + \lambda n^{-1/3})^l (1 + \lambda n^{-1/3})^k.$$

Using (1), if $k|\lambda|^3 \leq 3n$ then

$$(1 + \lambda n^{-1/3})^k = e^{k(\lambda n^{-1/3} - \lambda^2 n^{-2/3}/2 + O(\lambda^3 n^{-1}))} = e^{k(\lambda n^{-1/3} - \lambda^2 n^{-2/3}/2)} \left(1 + O\left(\frac{k\lambda^3}{n}\right)\right)$$

and similarly if $l|\lambda| \leq n^{1/3}$ then

$$(1 + \lambda n^{-1/3})^l = 1 + O(\lambda n^{-1/3}).$$

This gives the first part of the lemma; for the second, we use instead the fact that $\log(1+x) \leq x - x^2/2 + x^3/3$ for all x . □

Lemma 6. Suppose that $n \geq 4$, $k \geq 1$ and $|\lambda| \leq n^{1/3}/2$. If $k+l \leq 3n$ then

$$(1 - p_{n,\lambda})^{\binom{k}{2} - (k+l) + k(n-k)} = e^{-k - \lambda kn^{-1/3} + k^2/(2n) + \lambda k^2/(2n^{4/3})} (1 + O(\frac{k+l}{n})).$$

Furthermore, there exists a constant c such that for any $n \geq 4$, $k \geq 1$, $l \geq -1$ and $|\lambda| \leq n^{1/3}/2$,

$$(1 - p_{n,\lambda})^{\binom{k}{2} - (k+l) + k(n-k)} \leq ce^{-k - \lambda kn^{-1/3} + k^2/(2n) + \lambda k^2/(2n^{4/3})} (1 - 2/n)^{-l}.$$

Proof. We start by writing

$$(1 - p_{n,\lambda})^{\binom{k}{2} - (k+l) + k(n-k)} = (1 - p_{n,\lambda})^{kn - k^2/2} (1 - p_{n,\lambda})^{-3k/2 - l}$$

and treating the two terms separately. For the last term, if $k + l \leq 3n$ then

$$(1 - p_{n,\lambda})^{-3k/2 - l} = 1 + O\left(\frac{k+l}{n}\right),$$

and otherwise (since we always have $k \leq n$)

$$(1 - p_{n,\lambda})^{-3k/2 - l} = (1 - p_{n,\lambda})^{-l} (1 + O(k/n)) \leq c(1 - 2/n)^{-l}.$$

For the other term $(1 - p_{n,\lambda})^{kn - k^2/2}$, we use (1) to get

$$\begin{aligned} (1 - p_{n,\lambda})^{kn - k^2/2} &= e^{-kn p_{n,\lambda} + k^2 p_{n,\lambda}/2} (1 + O(kn p_{n,\lambda}^2) + O(k^2 p_{n,\lambda}^2)) \\ &= e^{-k - \lambda k n^{-1/3} + k^2/(2n) + \lambda k^2/(2n^{4/3})} (1 + O(\frac{k}{n})). \end{aligned}$$

Combining these estimates gives the result. \square

We can now complete the proof of Proposition 3 by combining the results above.

Proof of Proposition 3. We first concentrate on the first bound. For $k \leq n^{3/4}$, the first part of Lemma 4 holds, i.e.

$$\binom{n}{k} = \frac{n^k}{(2\pi)^{1/2} k^{k+1/2}} \exp\left(k - \frac{k^2}{2n} - \frac{k^3}{6n^2}\right) \left(1 + O\left(\frac{1}{k} + \frac{k}{n} + \frac{k^4}{n^3}\right)\right).$$

Also, when $k \leq n^{3/4}$, $l \leq 4n^{1/4}$ and $|\lambda| \leq n^{1/12}$, the first part of Lemma 5 holds, i.e.

$$p_{n,\lambda}^{k+l} = \frac{1}{n^{k+l}} e^{\lambda k n^{-1/3} - \frac{1}{2} \lambda^2 k n^{-2/3}} \left(1 + O\left(\frac{\lambda l}{n^{1/3}}\right) + O\left(\frac{k \lambda^3}{n}\right)\right).$$

Thirdly, when $k \leq n^{3/4}$ and $l \leq 4n^{1/4}$, the first part of Lemma 6 holds, i.e.

$$(1 - p_{n,\lambda})^{\binom{k}{2} - (k+l) + k(n-k)} = e^{-k - \lambda k n^{-1/3} + k^2/(2n) + \lambda k^2/(2n^{4/3})} \left(1 + O\left(\frac{k+l}{n}\right)\right).$$

As mentioned above (see (2)),

$$\mathbb{E}_{n,\lambda}[X(k, k+l)] = \binom{n}{k} C(k, k+l) p_{n,\lambda}^{k+l} (1 - p_{n,\lambda})^{\binom{k}{2} - (k+l) + k(n-k)}.$$

Combining the three approximations above gives the first part of the result. For the second part, we instead use the second parts of Lemmas 4, 5 and 6, together with the approximation

$$C(k, k+l) \leq c(l \vee 1)^{-l/2} k^{k+(3l-1)/2}$$

which holds for all l and k and is due to Bollobás [5, Corollary 5.21]. \square

3 Wright's coefficients

We can see from Proposition 3 that we need to know how $C(k, k+l)$ behaves. Cayley's formula tells us that $C(k, k-1) = k^{k-2}$. It is also well-known that $C(k, k) = (\pi/8)^{1/2} k^{k-1/2} (1 + O(k^{-1/2}))$; see [5, Corollary 5.19] for example. In this section we give details of asymptotics for $C(k, k+l)$ for other values of l . For a more thorough treatment we refer to Janson [13].

Wright's coefficients $(\gamma_l, l \geq 1)$ satisfy

$$\gamma_l = \frac{\pi^{1/2} 3^l (l-1)! d_l}{2^{(5l-1)/2} \Gamma(3l/2)} \quad (3)$$

where $d_1 = d_2 = 5/36$ and

$$d_{l+1} = d_l + \sum_{i=1}^{l-1} \frac{d_i d_{l-i}}{(l+1) \binom{l}{i}}, \quad l \geq 2.$$

Wright [22] gave an asymptotic for $C(k, k+l)$ in terms of γ_l for $l = o(k^{1/3})$. This was later improved by various authors, and in particular we now know that

$$C(k, k+l) = \gamma_l k^{k+(3l-1)/2} (1 + O(\frac{l^2}{k} + \frac{(l+1)^{1/16}}{k^{9/50}})) \quad \text{for all } l \leq 4k^{1/2}; \quad (4)$$

see [4]. See also [15] and [12] for two beautiful proofs, the former only a few pages long and using the Erdős-Rényi random graph, of slightly less precise asymptotics.

Clearly the sequence (d_l) is increasing; Wright [22, Theorem 3] showed that it is bounded above and therefore converges to some limit d , which Voblyi [21, Theorem 3] identified as $1/(2\pi)$. We can adapt Wright's proof that the sequence converges to bound the rate of convergence.

Lemma 7. *As $l \rightarrow \infty$, $1/(2\pi) - d_l = O(1/l)$.*

Proof. Since d_l is an increasing sequence, $d_l \leq d = 1/(2\pi)$ for all l . Therefore for any $j \geq 2$,

$$\begin{aligned} d_{j+1} - d_j &= \sum_{i=1}^{j-1} \frac{d_i d_{j-i}}{(j+1) \binom{j}{i}} \leq 2d^2 \sum_{i=1}^{\lfloor j/2 \rfloor} \frac{1}{(j+1) \binom{j}{i}} \\ &\leq 2d^2 \sum_{i=1}^{\lfloor j/2 \rfloor} \frac{i}{(j+1)j} \frac{(i-1)(i-2) \dots 1}{(j-1)(j-2) \dots (j-i)} \\ &\leq \frac{2d^2}{j(j+1)} \sum_{i=1}^{\lfloor j/2 \rfloor} i \left(\frac{i}{j}\right)^{i-1} \leq \frac{2d^2}{j(j+1)} \sum_{i=1}^{\infty} i \left(\frac{1}{2}\right)^{i-1} = \frac{8d^2}{j(j+1)}. \end{aligned}$$

Thus

$$1/(2\pi) - d_l = d - d_l = \sum_{j=l}^{\infty} (d_{j+1} - d_j) \leq 8d^2 \sum_{j=l}^{\infty} 1/j^2 = O(1/l). \quad \square$$

Applying Stirling's formula and then Lemma 7 to (3), we obtain for $l \geq 1$

$$\gamma_l = d_l (3\pi)^{1/2} \left(\frac{e}{12l}\right)^{l/2} (1 + O(1/l)) = \left(\frac{3}{4\pi}\right)^{1/2} \left(\frac{e}{12l}\right)^{l/2} (1 + O(1/l)). \quad (5)$$

We also know that $C(k, k-1) = k^{k-2}$ and $C(k, k) = (\pi/8)^{1/2} k^{k-1/2} (1 + O(k^{-1/2}))$, so in fact if we replace $\frac{e}{12l}$ with $\frac{e}{12(l \vee 1)}$ then (5) holds for all $l \geq -1$.

Combining this with Proposition 3 and (4), we immediately get the following corollary. The first line of the equality is more useful for small l , and the second line for larger l .

Corollary 8. *Suppose that $n \geq 16$, $|\lambda| \leq n^{1/12}$, $1 \leq k \leq n^{3/4}$ and $-1 \leq l \leq 4k^{1/2} \wedge 4n^{1/4}$. Then*

$$\begin{aligned} \mathbb{E}_{n,\lambda}[X(k, k+l)] &= \frac{\gamma_l k^{3l/2-1}}{(2\pi)^{1/2} n^l} e^{-F_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_3 + \frac{l^2}{k} + \frac{(l+1)^{1/16}}{k^{9/50}})) \\ &= \left(\frac{3}{8\pi}\right)^{1/2} \frac{1}{k} \left(\frac{ek^3}{12n^2(l \vee 1)}\right)^{l/2} e^{-F_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_3 + \frac{l^2}{k} + \frac{(l+1)^{1/16}}{k^{9/50}} + \frac{1}{l})). \end{aligned}$$

The following simple bound will be useful when k is small.

Corollary 9. *For any $M > 0$, there exist constants $0 < c_1 \leq c_2 < \infty$ depending on M such that*

$$c_1 k^{k-2} n \leq \sum_{l=-1}^L \frac{C(k, k+l)}{n^l} \leq c_2 k^{k-2} n$$

for all $n \geq 1$, $k \leq Mn^{2/3}$, and $L \leq 4k^{1/2}$.

Proof. By (4) and (5), we have

$$c_1 \left(\frac{e}{12(l \vee 1)} \right)^{l/2} k^{k+(3l-1)/2} \leq C(k, k+l) \leq c_2 \left(\frac{e}{12(l \vee 1)} \right)^{l/2} k^{k+(3l-1)/2}$$

for all $-1 \leq l \leq 4k^{1/2}$. Dividing by n^l and summing over $l \leq L$,

$$c_1 k^{k-1/2} \sum_{l=-1}^L \left(\frac{ek^3}{12n^2(l \vee 1)} \right)^{l/2} \leq \sum_{l=-1}^L \frac{C(k, k+l)}{n^l} \leq c_2 k^{k-1/2} \sum_{l=-1}^L \left(\frac{ek^3}{12n^2(l \vee 1)} \right)^{l/2}.$$

Since $k^3/n^2 \leq M^3$, the sum is bounded above and below by constants times its first term, which gives the result. \square

Inspecting the second line of Corollary 8, and letting $a = k/n^{2/3}$, we see that if we are interested in all components of size k when $k \approx an^{2/3}$ then we will need to estimate $\sum_l \left(\frac{ea^3}{12l} \right)^{l/2}$ for large a . The following lemma will be useful for this purpose.

Lemma 10. *If $x \in \left[\frac{a^3}{12}(1 - a^{-4/3}), \frac{a^3}{12}(1 + a^{-4/3}) \right]$, then*

$$\left(\frac{ea^3}{12x} \right)^{x/2} = \exp \left(\frac{a^3}{24} - \frac{a^3 y^2}{48} + O(a^{-1}) \right)$$

where $y = 12x/a^3 - 1$.

Proof. Let $y = 12x/a^3 - 1$, so that $x = \frac{a^3}{12}(1 + y)$. Then using (1),

$$\begin{aligned} \left(\frac{ea^3}{12x} \right)^{x/2} &= \left(\frac{e}{1+y} \right)^{a^3(1+y)/24} = e^{a^3(1+y)/24 - a^3(1+y) \log(1+y)/24} \\ &= e^{a^3(1+y)/24 - a^3(1+y)(y - y^2/2 + O(y^3))/24} \\ &= e^{a^3/24 - a^3 y^2/48 + O(a^3 y^3)}. \end{aligned}$$

Now, $|y| \leq a^{-4/3}$, so $|a^3 y^3| \leq 1/a$ and the result follows. \square

This allows us to bound the sum of the terms involving l in Proposition 3.

Lemma 11. *If $k \geq n^{2/3}$ and $k^3/n^2 \leq L \leq 4k^{1/2}$, then*

$$\sum_{l=-1}^L \frac{C(k, k+l)}{n^l} = \frac{k^{k+1}}{2n} \exp \left(\frac{k^3}{24n^2} \right) (1 + O(\frac{n^{2/3}}{k} + \frac{k^5}{n^4} + \frac{k^{3/400}}{n^{1/8}})).$$

Proof. Write $a = k/n^{2/3}$. When a is a constant, the $O(n^{2/3}/k)$ error means that we only need to bound above and below by constants times k^{k-2}/n , which we did in Corollary 9 (in this case k^{k+1}/n and $k^{k-2}n$ are of the same order). Therefore we may assume that a is large.

Note that $(\frac{ea^3}{12l})^{l/2}$ is increasing in l for $l \leq a^3/12$, and decreasing in l for $l \geq a^3/12$. Let $J^- = \lfloor \frac{a^3}{12}(1 - a^{-4/3}) \rfloor$ and $J^+ = \lfloor \frac{a^3}{12}(1 + a^{-4/3}) \rfloor$. Clearly

$$\sum_{l=J^-}^{J^+} \frac{C(k, k+l)}{n^l} \leq \sum_{l=-1}^L \frac{C(k, k+l)}{n^l} \leq \sum_{l=J^-}^{J^+} \frac{C(k, k+l)}{n^l} + \sum_{l=-1}^{J^-} \frac{C(k, k+l)}{n^l} + \sum_{l=J^+}^{\infty} \frac{C(k, k+l)}{n^l}.$$

We start by estimating $\sum_{l=J^-}^{J^+} \frac{C(k, k+l)}{n^l}$, and then show that the other two terms on the right-hand side are small.

By (4) and (5), for $l \in [J^-, J^+]$,

$$C(k, k+l) = \left(\frac{3}{4\pi}\right)^{1/2} k^{k-1/2} \left(\frac{ek^3}{12l}\right)^{l/2} \left(1 + O\left(\frac{n^2}{k^3} + \frac{k^5}{n^4} + \frac{k^{3/400}}{n^{1/8}}\right)\right).$$

Thus

$$\sum_{l=J^-}^{J^+} \frac{C(k, k+l)}{n^l} = (1 + O\left(\frac{n^2}{k^3} + \frac{k^5}{n^4} + \frac{k^{3/400}}{n^{1/8}}\right)) \left(\frac{3}{4\pi}\right)^{1/2} k^{k-1/2} \sum_{l=J^-}^{J^+} \left(\frac{ea^3}{12l}\right)^{l/2}. \quad (6)$$

Now, since $(\frac{ea^3}{12x})^{x/2}$ is increasing for $x < a^3/12$ and decreasing for $x > a^3/12$, by considering the two regions separately

$$\int_{J^-+1}^{J^+-1} \left(\frac{ea^3}{12x}\right)^{x/2} dx - 2e^{a^3/24} \leq \sum_{l=J^-+1}^{J^+-1} \left(\frac{ea^3}{12l}\right)^{l/2} \leq \int_{J^-}^{J^+} \left(\frac{ea^3}{12x}\right)^{x/2} dx + 2e^{a^3/24}.$$

But by Lemma 10,

$$\begin{aligned} \int_{J^-}^{J^+} \left(\frac{ea^3}{12x}\right)^{x/2} dx &= (1 + O(a^{-1})) \frac{a^3}{12} \int_{12J^-/a^3-1}^{12J^+/a^3-1} e^{a^3/24 - a^3 y^2/48} dy \\ &= (1 + O(a^{-1})) \frac{a^3}{12} e^{a^3/24} \int_{-\infty}^{\infty} e^{-a^3 y^2/48} dy \\ &= (1 + O(a^{-1})) \frac{a^3}{12} e^{a^3/24} (48\pi/a^3)^{1/2} \end{aligned}$$

and similarly for $\int_{J^-+1}^{J^+-1} (\frac{ea^3}{12x})^{x/2} dx$ (using the assumption, from the beginning of the proof, that a is large). Plugging this estimate into (6) gives

$$\sum_{l=J^-}^{J^+} \frac{C(k, k+l)}{n^l} = \frac{k^{k+1}}{2n} \exp\left(\frac{k^3}{24n^2}\right) (1 + O\left(\frac{n^{2/3}}{k} + \frac{k^5}{n^4} + \frac{k^{3/400}}{n^{1/8}}\right)),$$

so it now suffices to show that both $\sum_{l=1}^{J^-} \frac{C(k, k+l)}{n^l}$ and $\sum_{l=J^+}^{\infty} \frac{C(k, k+l)}{n^l}$ are relatively small.

By Lemma 10,

$$\sum_{l=1}^{J^-} \left(\frac{ea^3}{12l}\right)^{l/2} \leq \frac{a^3}{12} \exp\left(\frac{a^3}{24} - \frac{a^{1/3}}{48} + O(a^{-1})\right).$$

Similarly,

$$\sum_{l=J^+}^{\lfloor a^3 \rfloor} \left(\frac{ea^3}{12l}\right)^{l/2} \leq a^3 \exp\left(\frac{a^3}{24} - \frac{a^{1/3}}{48} + O(a^{-1})\right),$$

and trivially

$$\sum_{l=\lfloor a^3 \rfloor}^{\infty} \left(\frac{e}{12}\right)^{l/2} \leq \frac{1}{2} e^{-a^3/2}.$$

Combining these bounds with (4) and (5), we get the result. \square

4 More first moment asymptotics

For $k \in \mathbb{N}$, let $Y(k)$ be the number of components of size exactly k , and for $k \leq n^{3/4}$, let $Z(k)$ the number of components of size between k and $n^{3/4}$. In this section we give first moment asymptotics for $Y(k)$ and $Z(k)$. Recall that

$$G_\lambda(x) = \frac{x^3}{8} - \frac{\lambda x^2}{2} + \frac{\lambda^2 x}{2} = \frac{x}{8} (x - 2\lambda)^2 = F_\lambda(x) - \frac{x^3}{24}.$$

Recall also the definitions of \mathcal{E}_1 and \mathcal{E}_2 from Section 1. The main two results in this section are the following.

Proposition 12. *Suppose that $n \geq 25$, $|\lambda| \leq n^{1/12}$ and $n^{2/3} \leq k \leq n^{3/4}$. Then*

$$\mathbb{E}_{n,\lambda}[Y(k)] = \frac{k^{1/2}}{(8\pi)^{1/2}n} e^{-G_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_1)).$$

Proposition 13. *Suppose that $n \geq 25$, $|\lambda| \leq n^{1/12}$ and $(3\lambda \vee 1)n^{2/3} \leq k \leq n^{4/3}$. Then*

$$\mathbb{E}_{n,\lambda}[Z(k)] = \frac{1}{(8\pi)^{1/2}} \frac{(k/n^{2/3})^{1/2}}{G'_\lambda(k/n^{2/3})} e^{-G_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_1)) + O(\mathcal{E}_2).$$

To prove Proposition 12, we write

$$\mathbb{E}_{n,\lambda}[Y(k)] = \sum_{l \geq -1} \mathbb{E}_{n,\lambda}[X(k, k+l)].$$

We will use Proposition 3 and Lemma 11 to bound the sum for small l , but first we need a bound on the sum for large l .

Lemma 14. *There exists a finite constant c such that if $n \geq 25$, $|\lambda| \leq n^{1/12}$, $1 \leq k \leq n^{3/4}$ and $L \geq 4n^{1/4} \wedge 4k^{1/3}$,*

$$\sum_{l=L}^{\infty} \mathbb{E}_{n,\lambda}[X(k, k+l)] \leq ce^{-L/2}.$$

Proof. First we claim that

$$\sum_{l=L}^{\infty} \left(\frac{k^3}{n^2 l}\right)^{l/2} \left(\frac{1 + \lambda n^{-1/3}}{1 - 2n^{-1}}\right)^l \leq 3e^{-L/2}.$$

It is easy to check that when $\lambda \leq n^{1/12}$ and $n \geq 25$, we have $\frac{1 + \lambda n^{-1/3}}{1 - 2n^{-1}} \leq 2e^{-1/2}$; and also that when $l \geq L$ and $k \leq n^{3/4}$, we have $k^3/(n^2 l) \leq 1/4$. Therefore

$$\sum_{l=L}^{\infty} \left(\frac{k^3}{n^2 l}\right)^{l/2} \left(\frac{1 + \lambda n^{-1/3}}{1 - 2n^{-1}}\right)^l \leq \sum_{l=L}^{\infty} e^{-l/2} \leq 3e^{-L/2}$$

as claimed. But the second part of Proposition 3 tells us that

$$\mathbb{E}_{n,\lambda}[X(k, k+l)] \leq \frac{c}{k} \left(\frac{k^3}{n^2 l}\right)^{l/2} e^{-F_\lambda(k/n^{2/3}) + \frac{1}{3}\lambda^3 k n^{-1}} \left(\frac{1 + \lambda n^{-1/3}}{1 - 2n^{-1}}\right)^l.$$

It is easy to check that $F_\lambda(x) \geq 0$ for all $x \geq 0$, and since $|\lambda| \leq 1/12$ and $k \leq n^{3/4}$ we have $\lambda^3 k n^{-1} \leq 1$ so we get the result. \square

Proof of Proposition 12. Proposition 3 tells us that for $l \leq 4n^{1/4}$,

$$\mathbb{E}_{n,\lambda}[X(k, k+l)] = \frac{C(k, k+l)}{(2\pi)^{1/2} n^l k^{k+1/2}} e^{-F_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_3(k, n, l, \lambda))).$$

Applying Lemma 11 with $L = 4n^{1/4}$ (which is larger than k^3/n^2 since $k \leq n^{3/4}$) gives

$$\sum_{l=-1}^{\lfloor 4n^{1/4} \rfloor} \frac{C(k, k+l)}{n^l} = \frac{k^{k+1}}{2n} e^{k^3/(24n^2)} (1 + O(\frac{n^{2/3}}{k} + \frac{k^5}{n^4} + \frac{k^{3/400}}{n^{1/8}})).$$

Combining these and simplifying the error terms, we get

$$\sum_{l=-1}^{\lfloor 4n^{1/4} \rfloor} \mathbb{E}_{n,\lambda}[X(k, k+l)] = \frac{k^{1/2}}{(8\pi)^{1/2}n} e^{-G_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_1)).$$

Finally, Lemma 14 gives

$$\sum_{l=\lceil 4n^{1/4} \rceil}^{\infty} \mathbb{E}_{n,\lambda}[X(k, k+l)] \leq ce^{-2n^{1/4}}$$

which can be absorbed into the $O(\mathcal{E}_1)$ term. \square

We now move on to proving Proposition 13. We will sum $\mathbb{E}_{n,\lambda}[Y(j)]$ from k to $\lfloor n^{3/4} \rfloor$; we simply need to estimate the resulting sum, which we do in the next two lemmas.

Lemma 15. *For $a \geq 1 \vee 3\lambda$ and any constant $r \geq 0$,*

$$\int_a^\infty y^r e^{-G_\lambda(y)} dy = \frac{a^r}{G'_\lambda(a)} e^{-G_\lambda(a)} (1 + O(\frac{1}{a^3})).$$

Proof. Writing $y^r e^{-G_\lambda(y)}$ as $(y^r/G'_\lambda(y)) \cdot G'_\lambda(y) e^{-G_\lambda(y)}$ and integrating by parts, we get

$$\int_a^\infty y^r e^{-G_\lambda(y)} dy = \frac{a^r}{G'_\lambda(a)} e^{-G_\lambda(a)} + \int_a^\infty \left(\frac{r}{yG'_\lambda(y)} - \frac{G''_\lambda(y)}{G'_\lambda(y)^2} \right) y^r e^{-G_\lambda(y)} dy.$$

It is then straightforward to check that for $y \geq a$, provided that $\lambda \leq a/3$,

$$\frac{r}{yG'_\lambda(y)} - \frac{G''_\lambda(y)}{G'_\lambda(y)^2} = O\left(\frac{1}{a^3}\right).$$

Thus

$$\left(1 + O\left(\frac{1}{a^3}\right)\right) \int_a^\infty y^r e^{-G_\lambda(y)} dy = \frac{a^r}{G'_\lambda(a)} e^{-G_\lambda(a)}$$

and the result follows. \square

Lemma 16. *For $(1 \vee 3\lambda)n^{2/3} \leq k \leq n^{3/4}$ and $|\lambda| \leq n^{1/12}$, and any constant $r \geq 0$,*

$$\frac{1}{n^{2/3}} \sum_{j=k}^{\lfloor n^{3/4} \rfloor} \left(\frac{j}{n^{2/3}}\right)^r e^{-G_\lambda(j/n^{2/3})} = \frac{(k/n^{2/3})^r}{G'_\lambda(k/n^{2/3})} e^{-G_\lambda(k/n^{2/3})} (1 + O(\frac{n^2}{k^3})) + O(n^{(r-2)/12} e^{-G_\lambda(n^{1/12})}).$$

Proof. Write $N = \lfloor n^{3/4} \rfloor$ and

$$S = \frac{1}{n^{2/3}} \sum_{j=k}^N \left(\frac{j}{n^{2/3}}\right)^r e^{-G_\lambda(j/n^{2/3})}.$$

For a lower bound, since x^r is increasing in x and $e^{-G_\lambda(x/n^{2/3})}$ is decreasing in x for $x \geq 2\lambda n^{2/3}$,

$$S \geq \frac{1}{n^{2/3}} \int_k^N \left(\frac{x-1}{n^{2/3}}\right)^r e^{-G_\lambda(x/n^{2/3})} dx.$$

Now $(x-1)^r = x^r(1 + O(1/x))$, so

$$S \geq (1 + O(\frac{1}{k})) \frac{1}{n^{2/3}} \int_k^N \left(\frac{x}{n^{2/3}}\right)^r e^{-G_\lambda(x/n^{2/3})} dx.$$

Substituting $y = x/n^{2/3}$, writing $a = k/n^{2/3}$ and applying Lemma 15, we get

$$S \geq \frac{a^r}{G'_\lambda(a)} e^{-G_\lambda(a)} (1 + O(\frac{1}{a^3} + \frac{1}{k})) - \frac{n^{r/12}}{G'_\lambda(n^{1/12})} e^{-G_\lambda(n^{1/12})} (1 + O(1)).$$

Noting that $G'_\lambda(x)$ is of order x^2 for all $x \geq 1 \vee 3\lambda$ gives the desired lower bound ($1/k \leq 1/a^3$ so we may ignore that error term). We proceed similarly for the upper bound. Again since x^r is increasing in x and $e^{-G_\lambda(x/n^{2/3})}$ is decreasing in x for $x \geq 2\lambda n^{2/3}$,

$$\begin{aligned} S &\leq \frac{1}{n^{2/3}} \left(\frac{k}{n^{2/3}} \right)^r e^{-G_\lambda(k/n^{2/3})} + \frac{1}{n^{2/3}} \int_{k+1}^{N+1} \left(\frac{x}{n^{2/3}} \right)^r e^{-G_\lambda((x-1)/n^{2/3})} dx \\ &= \frac{1}{n^{2/3}} \left(\frac{k}{n^{2/3}} \right)^r e^{-G_\lambda(k/n^{2/3})} + \frac{1}{n^{2/3}} \int_k^N \left(\frac{x+1}{n^{2/3}} \right)^r e^{-G_\lambda(x/n^{2/3})} dx \\ &= \frac{1}{n^{2/3}} \left(\frac{k}{n^{2/3}} \right)^r e^{-G_\lambda(k/n^{2/3})} + (1 + O(\frac{1}{k})) \frac{1}{n^{2/3}} \int_k^N \left(\frac{x}{n^{2/3}} \right)^r e^{-G_\lambda(x/n^{2/3})} dx. \end{aligned}$$

Now again substituting $y = x/n^{2/3}$, writing $a = k/n^{2/3}$ and applying Lemma 15, we have

$$\frac{1}{n^{2/3}} \int_k^N \left(\frac{x}{n^{2/3}} \right)^r e^{-G_\lambda(x/n^{2/3})} dx = \frac{a^r}{G'_\lambda(a)} e^{-G_\lambda(a)} (1 + O(\frac{1}{a^3})) - \frac{n^{r/12}}{G'_\lambda(n^{1/12})} e^{-G_\lambda(n^{1/12})} (1 + O(1)).$$

Again we use the fact that $G'_\lambda(x)$ is of order x^2 for $x \geq 1 \vee 3\lambda$, which means that both

$$\frac{n^{r/12}}{G'_\lambda(n^{1/12})} = O(n^{(r-2)/12}) \quad \text{and} \quad \frac{1}{n^{2/3}} = O\left(\frac{k^2}{n^3}\right) \frac{1}{G'_\lambda(a)}.$$

These estimates give us

$$S \leq O\left(\frac{k^2}{n^3}\right) \frac{a^r}{G'_\lambda(a)} e^{-G_\lambda(a)} + \frac{a^r}{G'_\lambda(a)} e^{-G_\lambda(a)} (1 + O(\frac{1}{a^3})) + O(n^{(r-2)/12} e^{-G_\lambda(n^{1/12})}).$$

Noting that $k^2/n^3 \leq 1/a^3$, we can drop that error term and the proof is complete. \square

Proof of Proposition 13. Recall that $Z(k)$ is the number of components of size between k and $n^{3/4}$. By Proposition 12,

$$\mathbb{E}_{n,\lambda}[Z(k)] = \sum_{j=k}^{\lfloor n^{3/4} \rfloor} \mathbb{E}_{n,\lambda}[Y(j)] = \sum_{j=k}^{\lfloor n^{3/4} \rfloor} \frac{j^{1/2}}{(8\pi)^{1/2} n} e^{-G_\lambda(j/n^{2/3})} (1 + O(\mathcal{E}_1(j, n, \lambda))).$$

By Lemma 16 (carefully summing the terms involving j in $\mathcal{E}_1(j, n, \lambda)$ by changing the value of r used in Lemma 16), this is

$$\frac{(k/n^{2/3})^{1/2}}{G'_\lambda(k/n^{2/3})} e^{-G_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_1(k, n, \lambda))) + O(n^{-1/8} e^{-G_\lambda(n^{1/12})}).$$

It is easily checked that since $\lambda \leq n^{1/12}/3$, we have $G_\lambda(n^{1/12}) \geq n^{1/4}/72$ so the last error is of order smaller than \mathcal{E}_2 . \square

5 Second moment bounds

We now have the bounds that we want on $\mathbb{E}_{n,\lambda}[X(k, k+l)]$, $\mathbb{E}_{n,\lambda}[Y(k)]$ and $\mathbb{E}_{n,\lambda}[Z(k)]$. To show that these expectations give asymptotically tight bounds on $\mathbb{P}_{n,\lambda}(X(k, k+l) \geq 1)$, $\mathbb{P}_{n,\lambda}(Y(k) \geq 1)$, and $\mathbb{P}_{n,\lambda}(Z(k) \geq 1)$, we need to develop some relatively easy second moment bounds.

Lemma 17. *If $k \leq n^{3/4}$ and $n \geq 81$, then*

$$\mathbb{E}_{n,\lambda}[X(k, k+l)^2] = \mathbb{E}_{n,\lambda}[X(k, k+l)] + \mathbb{E}_{n,\lambda}[X(k, k+l)]^2 \exp\left(\frac{\lambda k^2}{n^{4/3}} - \frac{k^3}{n^2}\right) (1 + O(\frac{k}{n} + \frac{k^4}{n^3})).$$

Also, if $j, k \leq n^{3/4}$, $n \geq 81$ and either $j \neq k$ or $l \neq l'$, then

$$\begin{aligned} \mathbb{E}_{n,\lambda}[X(j, j+l)X(k, k+l')] \\ = \mathbb{E}_{n,\lambda}[X(j, j+l)]\mathbb{E}_{n,\lambda}[X(k, k+l')] \exp\left(\frac{\lambda jk}{n^{4/3}} - \frac{jk}{2n^2}(j+k)\right) (1 + O(\frac{j+k}{n} + \frac{j^4+k^4}{n^3})). \end{aligned}$$

Proof. Let $\mathcal{C}(k, k+l)$ be the set of components with k vertices and $k+l$ edges. First note that

$$\mathbb{E}_{n,\lambda}[X(k, k+l)^2] = \mathbb{E}_{n,\lambda}\left[\sum_{|S|=k} \sum_{|S'|=k} \mathbb{1}_{\{S \in \mathcal{C}(k, k+l)\}} \mathbb{1}_{\{S' \in \mathcal{C}(k, k+l)\}}\right].$$

If S is a component, then S and S^c are not connected by an edge; therefore if S and S' are components with $S \cap S' \neq \emptyset$, then necessarily $S = S'$. Thus

$$\mathbb{E}_{n,\lambda}[X(k, k+l)^2] = \mathbb{E}_{n,\lambda}\left[\sum_{|S|=k} \mathbb{1}_{\{S \in \mathcal{C}(k, k+l)\}}\right] + \mathbb{E}_{n,\lambda}\left[\sum_{|S|=k} \sum_{\substack{|S'|=k, \\ S' \cap S = \emptyset}} \mathbb{1}_{\{S \in \mathcal{C}(k, k+l)\}} \mathbb{1}_{\{S' \in \mathcal{C}(k, k+l)\}}\right]$$

and if either $j \neq k$ or $l \neq l'$,

$$\mathbb{E}_{n,\lambda}[X(j, j+l)X(k, k+l')] = \mathbb{E}_{n,\lambda}\left[\sum_{|S|=j} \sum_{\substack{|S'|=k, \\ S' \cap S = \emptyset}} \mathbb{1}_{\{S \in \mathcal{C}(j, j+l)\}} \mathbb{1}_{\{S' \in \mathcal{C}(k, k+l')\}}\right].$$

Note that this last quantity equals (for any j, k, l, l')

$$\binom{n}{j} \binom{n-j}{k} C(j, j+l) C(k, k+l') p_{n,\lambda}^{j+l+k+l'} (1 - p_{n,\lambda})^{\binom{j}{2} + \binom{k}{2} - j - l - k - l' + j(n-j) + k(n-k)}.$$

By comparing with the formula (2) for $\mathbb{E}_{n,\lambda}[X(k, k+l)]$, we see that

$$\mathbb{E}_{n,\lambda}[X(k, k+l)^2] = \mathbb{E}_{n,\lambda}[X(k, k+l)] + \mathbb{E}_{n,\lambda}[X(k, k+l)]^2 \frac{\binom{n-k}{k}}{\binom{n}{k}} (1 - p_{n,\lambda})^{-k^2}$$

and if either $j \neq k$ or $l \neq l'$, then

$$\mathbb{E}_{n,\lambda}[X(j, j+l)X(k, k+l')] = \mathbb{E}_{n,\lambda}[X(j, j+l)]\mathbb{E}_{n,\lambda}[X(k, k+l')] \frac{\binom{n-j}{k}}{\binom{n}{k}} (1 - p_{n,\lambda})^{-jk}.$$

Therefore it suffices to show that for any $j, k \leq n^{3/4} \wedge \frac{n}{3}$ and $n \geq 81$,

$$\frac{\binom{n-j}{k}}{\binom{n}{k}} (1 - p_{n,\lambda})^{-jk} = \exp\left(\frac{\lambda jk}{n^{4/3}} - \frac{jk}{2n^2}(j+k)\right) (1 + O(\frac{j+k}{n} + \frac{j^4+k^4}{n^3})). \quad (7)$$

Since $n \geq 81$ and $j, k \leq n^{3/4}$ we also have $j, k \leq n/3$. Stirling's formula tells us that

$$\begin{aligned} \frac{\binom{n-j}{k}}{\binom{n}{k}} &= \frac{(n-j)!(n-k)!}{(n-j-k)!n!} = \frac{(n-j)^{n-j+1/2}(n-k)^{n-k+1/2}}{n^{n+1/2}(n-j-k)^{n-j-k+1/2}} (1 + O(1/n)) \\ &= \left(1 - \frac{j}{n}\right)^{n-j} \left(1 + \frac{j}{n-j-k}\right)^{n-j-k} \left(1 - \frac{k}{n}\right)^j (1 + O(\frac{j+k}{n})). \end{aligned}$$

Using the expansion (1),

$$\begin{aligned} \left(1 - \frac{j}{n}\right)^{n-j} &= \exp\left(-j + \frac{j^2}{2n} + \frac{j^3}{6n^2} + O(j^4/n^3)\right), \\ \left(1 + \frac{j}{n-j-k}\right)^{n-j-k} &= \exp\left(j - \frac{j^2}{2(n-j-k)} + \frac{j^3}{3(n-j-k)^2} + O\left(\frac{j^4}{n^3}\right)\right) \\ &= \exp\left(j - \frac{j^2}{2n} - \frac{j^2}{2n^2}(j+k) + \frac{j^3}{3n^2} + O\left(\frac{j^4}{n^3}\right)\right), \end{aligned}$$

and

$$\left(1 - \frac{k}{n}\right)^j = \exp\left(-\frac{jk}{n} - \frac{jk^2}{2n^2} + O\left(\frac{jk^3}{n^3}\right)\right).$$

Combining these, we get

$$\frac{\binom{n-j}{k}}{\binom{n}{k}} = \exp\left(-\frac{jk}{n} - \frac{jk}{2n^2}(j+k)\right)(1 + O\left(\frac{j+k}{n} + \frac{j^4+k^4}{n^3}\right)).$$

On the other hand,

$$(1 - p_{n,\lambda})^{-jk} = \exp\left(\frac{jk}{n} + \frac{\lambda jk}{n^{4/3}} + O\left(\frac{jk}{n^2}\right)\right).$$

These two approximations establish (7) and complete the proof. \square

For $k \in \mathbb{N}$, recall that we defined $Y(k)$ to be the number of components of size exactly k , and for $k \leq n^{3/4}$, we let $Z(k)$ be the number of components of size between k and $n^{8/11}/2$. We can use Lemma 17 to bound the second moments of $Y(k)$ and $Z(k)$.

Lemma 18. *If $k \leq n^{3/4}$ and $n \geq 81$, then*

$$\mathbb{E}_{n,\lambda}[Y(k)^2] = \mathbb{E}_{n,\lambda}[Y(k)] + \mathbb{E}_{n,\lambda}[Y(k)]^2 \exp\left(\frac{\lambda k^2}{n^{4/3}} - \frac{k^3}{n^2}\right)(1 + O\left(\frac{k}{n} + \frac{k^4}{n^3}\right))$$

and for $j \neq k$ with $j \leq n^{3/4}$,

$$\mathbb{E}_{n,\lambda}[Y(j)Y(k)] = \mathbb{E}_{n,\lambda}[Y(j)]\mathbb{E}_{n,\lambda}[Y(k)] \exp\left(\frac{\lambda jk}{n^{4/3}} - \frac{jk}{2n^2}(j+k)\right)(1 + O\left(\frac{j+k}{n} + \frac{j^4+k^4}{n^3}\right)).$$

Further, if $\lambda n^{2/3} \leq k \leq n^{3/4}$ and $n \geq 81$, then

$$\mathbb{E}_{n,\lambda}[Z(k)^2] \leq \mathbb{E}_{n,\lambda}[Z(k)] + \mathbb{E}_{n,\lambda}[Z(k)]^2 \exp\left(\frac{\lambda k^2}{n^{4/3}} - \frac{k^3}{n^2}\right)(1 + O(n^{-1/11})).$$

Proof. We begin with $\mathbb{E}_{n,\lambda}[Y(k)^2]$. Clearly

$$\begin{aligned} \mathbb{E}_{n,\lambda}[Y(k)^2] &= \sum_{l,l' \geq -1} \mathbb{E}_{n,\lambda}[X(k, k+l)X(k, k+l')] \\ &= \sum_{l \geq -1} \mathbb{E}_{n,\lambda}[X(k, k+l)^2] + \sum_{l \geq -1} \sum_{\substack{l' \geq -1, \\ l' \neq l}} \mathbb{E}_{n,\lambda}[X(k, k+l)X(k, k+l')]. \end{aligned}$$

By Lemma 17, this equals

$$\sum_{l \geq -1} \mathbb{E}_{n,\lambda}[X(k, k+l)] + \sum_{l,l' \geq -1} \mathbb{E}_{n,\lambda}[X(k, k+l)]\mathbb{E}_{n,\lambda}[X(k, k+l')] \exp\left(\frac{\lambda k^2}{n^{4/3}} - \frac{k^3}{n^2}\right)(1 + O\left(\frac{k}{n} + \frac{k^4}{n^3}\right)).$$

We now recognise $\mathbb{E}_{n,\lambda}[Y(k)] = \sum_{l \geq -1} \mathbb{E}_{n,\lambda}[X(k, k+l)]$ and the first part of the lemma follows.

Similarly, if $j \neq k$,

$$\begin{aligned}
& \mathbb{E}_{n,\lambda}[Y(j)Y(k)] \\
&= \sum_{l,l' \geq -1} \mathbb{E}_{n,\lambda}[X(j, j+l)X(k, k+l')] \\
&= \sum_{l,l'} \mathbb{E}_{n,\lambda}[X(j, j+l)] \mathbb{E}_{n,\lambda}[X(k, k+l')] e^{\lambda jk/n^{4/3} - jk(j+k)/(2n^2)} (1 + O(\frac{j+k}{n} + \frac{j^4+k^4}{n^3})) \\
&= \mathbb{E}_{n,\lambda}[Y(j)] \mathbb{E}_{n,\lambda}[Y(k)] e^{\lambda jk/n^{4/3} - jk(j+k)/(2n^2)} (1 + O(\frac{j+k}{n} + \frac{j^4+k^4}{n^3})).
\end{aligned}$$

This establishes the second part of the lemma.

For $\mathbb{E}_{n,\lambda}[Z(k)^2]$, we let $N = \lfloor n^{3/4} \rfloor$ and write

$$\begin{aligned}
\mathbb{E}_{n,\lambda}[Z(k)^2] &= \sum_{i=k}^N \sum_{j=k}^N \mathbb{E}_{n,\lambda}[Y(i)Y(j)] \\
&= \sum_{i=k}^N \mathbb{E}[Y(i)^2] + \sum_{i=k}^N \sum_{\substack{j=k, \dots, N, \\ j \neq i}} \mathbb{E}_{n,\lambda}[Y(i)Y(j)].
\end{aligned}$$

By the first two parts of the lemma, this equals

$$\sum_{i=k}^N \mathbb{E}[Y(i)^2] + \sum_{i,j=k}^N \mathbb{E}[Y(i)] \mathbb{E}[Y(j)] \exp\left(\frac{\lambda ij}{n^{4/3}} - \frac{ij}{2n^2}(i+j)\right) (1 + O(\frac{i+j}{n} + \frac{i^4+j^4}{n^3})).$$

Noting that if $\lambda n^{2/3} \leq k$ then the exponent above is decreasing in i and j for $i, j \geq k$, we get

$$\mathbb{E}_{n,\lambda}[Z(k)^2] \leq \sum_{i=k}^N \mathbb{E}[Y(i)^2] + \sum_{i,j=k}^N \mathbb{E}[Y(i)] \mathbb{E}[Y(j)] \exp\left(\frac{\lambda k^2}{n^{4/3}} - \frac{k^3}{n^2}\right) (1 + O(\frac{N}{n} + \frac{N^4}{n^3})).$$

Again recognising $\mathbb{E}_{n,\lambda}[Z(k)] = \sum_{i=k}^N \mathbb{E}[Y(i)]$, we get the desired result. \square

6 Large k : a simple exploration process argument

When k is large (say $k \geq n^{3/4}$) it is easiest to use a different approach to bounding the probability that there is a component of size at least k . We do not aim to give best possible bounds, and instead extract an easy argument from [17]. An improved bound (in particular not including the factor of $n^{1/4}$) could easily be obtained by more closely following the proof in [17].

Lemma 19. *If $\lambda \leq n^{1/12}/5$, then*

$$\mathbb{P}_{n,\lambda}(L_1 > k) \leq n^{1/4} \exp\left(-\frac{k^3}{80n^2}\right).$$

Proof. We use the following exploration process: start with one vertex, v , in a queue. At each step, choose a vertex from the queue, remove it from the queue, and add to the queue all of its neighbours that have not previously been in the queue. Stop when the queue is empty; at this stage the set of all vertices that have been in the queue at any step is exactly the set of vertices of $\mathcal{C}(v)$, the connected component containing v .

Let η_j be the number of vertices added to the queue at step j , so $\eta_j - 1$ is the change in the length of the queue at step j . Clearly η_j is stochastically dominated by a binomial random variable with parameters $(n - j, p_{n,\lambda})$. Let $Z_k = \sum_{j=1}^k (\eta_j - 1)$ and $T = \min\{k \geq 1 : Z_k = 0\}$. Then T is

the first step at which the queue is empty, and since we removed one vertex at each step, we must have $|\mathcal{C}(v)| = T$. Thus for any $\mu > 0$,

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| > k) \leq \mathbb{P}_{n,\lambda}(Z_k \geq 1) = \mathbb{P}_{n,\lambda}(e^{\mu Z_k} \geq e^\mu) \leq \mathbb{E}_{n,\lambda}[e^{\mu Z_k}]. \quad (8)$$

Since η_j is stochastically dominated by a binomial random variable with parameters $(n - j, p_{n,\lambda})$, for any $\mu > 0$ we have

$$\mathbb{E}_{n,\lambda}[e^{\mu(\eta_j-1)}] \leq e^{-\mu}(1 + (e^\mu - 1)p_{n,\lambda})^{n-j}.$$

Now for $\mu \leq 1$, $e^\mu - 1 \leq \mu + \mu^2$, and for any x , $1 + x \leq e^x$. Therefore for $\mu \leq 1$ the above is at most

$$e^{-\mu} e^{(n-j)\mu p_{n,\lambda} + (n-j)\mu^2 p_{n,\lambda}} \leq \exp(-\mu(1 - np_{n,\lambda}) - j\mu p_{n,\lambda} + \mu^2 np_{n,\lambda}).$$

Recalling that $p_{n,\lambda} = \frac{1}{n}(1 + \frac{\lambda}{n^{1/3}})$, we obtain

$$\mathbb{E}_{n,\lambda}[e^{\mu(\eta_j-1)}] \leq \exp\left(\frac{\mu\lambda}{n^{1/3}} - j\mu p_{n,\lambda} + \mu^2 np_{n,\lambda}\right).$$

Substituting this estimate into (8), we have that for any $\mu \in (0, 1]$,

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| > k) \leq \prod_{j=1}^k \exp\left(\frac{\mu\lambda}{n^{1/3}} - j\mu p_{n,\lambda} + \mu^2 np_{n,\lambda}\right) \leq \exp\left(\frac{k\mu\lambda}{n^{1/3}} - \frac{k^2}{2}\mu p_{n,\lambda} + k\mu^2 np_{n,\lambda}\right).$$

Choosing $\mu = k/(4n)$, we get

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| > k) \leq \exp\left(\frac{k^2\lambda}{4n^{4/3}} - \frac{k^3 p_{n,\lambda}}{8n} + \frac{k^3 p_{n,\lambda}}{16n}\right) = \exp\left(-\frac{k^3}{16n^2}\left(1 + \frac{\lambda}{n^{1/3}} - \frac{4n^{2/3}\lambda}{k}\right)\right).$$

If $k \geq n^{3/4}$ and $\lambda \leq n^{1/12}/5$ then $4n^{2/3}\lambda/k \leq 4/5$, so finally

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| > k) \leq \exp\left(-\frac{k^3}{16n^2}\left(\frac{1}{5} + \frac{\lambda}{n^{1/3}}\right)\right).$$

Clearly the probability on the left is increasing in λ , and the right-hand side is decreasing in λ , so without loss of generality we may take $\lambda = 0$. Finally,

$$\begin{aligned} \mathbb{P}_{n,\lambda}(L_1 > k) &\leq \mathbb{P}_{n,\lambda}(\#\{u : |\mathcal{C}(u)| > k\} > k) \leq \frac{1}{k} \mathbb{E}_{n,\lambda}[\#\{u : |\mathcal{C}(u)| > k\}] \\ &= \frac{n}{k} \mathbb{P}(|\mathcal{C}(v)| > k) \leq n^{1/4} \exp\left(-\frac{k^3}{80n^2}\right). \quad \square \end{aligned}$$

7 Proof of Theorem 1: moment bounds to probabilities

In this section we put together the moment estimates that we proved in previous sections to complete the proof of Theorem 1. We will at several points use the inequality

$$\mathbb{P}(V \geq 1) \geq \frac{\mathbb{E}[V]^2}{\mathbb{E}[V^2]} \quad (9)$$

which holds for any non-negative integer-valued random variable V and is easily proved by applying the Cauchy-Schwarz inequality to $\mathbb{E}[V \mathbb{1}_{\{V \geq 1\}}]$. Define

$$\mathcal{E}_4 = \mathcal{E}_4(k, n, \lambda) = \frac{n}{k^{3/2}} e^{-G_\lambda(k/n^{2/3}) + \lambda k^2/n^{4/3} - k^3/n^2}. \quad (10)$$

We now give two lemmas that relate the probabilities of events that appeared in Theorem 1 with the expectations that we calculated in Propositions 12 and 13.

Lemma 20. Suppose that $n \geq 81$, $-n^{1/12} \leq \lambda \leq n^{1/12}/5$, and $(3\lambda \wedge 1)n^{2/3} \leq k \leq n^{3/4}$. Then

$$\mathbb{P}_{n,\lambda}(L_1 = k) = \mathbb{E}_{n,\lambda}[Y(k)](1 + O(\mathcal{E}_4)) + O(\mathcal{E}_2).$$

Proof. Clearly $\mathbb{P}_{n,\lambda}(L_1 = k) \leq \mathbb{P}_{n,\lambda}(Y(k) \geq 1) \leq \mathbb{E}_{n,\lambda}[Y(k)]$, so we may concentrate on the lower bound. Let $N = \lfloor n^{3/4} \rfloor$. We have

$$\begin{aligned} \mathbb{P}(L_1 = k) &= \mathbb{P}(Y(k) \geq 1) - \mathbb{P}(Y(k) \geq 1, \exists j \geq k \text{ such that } Y(j) \geq 1) \\ &\geq \mathbb{P}(Y(k) \geq 1) - \sum_{j=k+1}^N \mathbb{P}(Y(k) \geq 1, Y(j) \geq 1) - \mathbb{P}(L_1 > N) \\ &\geq \frac{\mathbb{E}[Y(k)]^2}{\mathbb{E}[Y(k)^2]} - \sum_{j=k+1}^N \mathbb{E}[Y(k)Y(j)] - \mathbb{P}(L_1 > N) \end{aligned}$$

where for the last line we have applied (9) and Markov's inequality.

By Lemma 18 we have

$$\mathbb{E}_{n,\lambda}[Y(k)^2] = \mathbb{E}_{n,\lambda}[Y(k)](1 + O(\mathbb{E}_{n,\lambda}[Y(k)]e^{\lambda k^2/n^{4/3} - k^3/n^2})),$$

so

$$\frac{\mathbb{E}[Y(k)]^2}{\mathbb{E}[Y(k)^2]} = \mathbb{E}_{n,\lambda}[Y(k)](1 + O(\mathbb{E}_{n,\lambda}[Y(k)]e^{\lambda k^2/n^{4/3} - k^3/n^2})).$$

Lemma 18 also tells us that for $j > k$,

$$\mathbb{E}_{n,\lambda}[Y(k)Y(j)] = \mathbb{E}_{n,\lambda}[Y(k)]\mathbb{E}_{n,\lambda}[Y(j)]e^{\lambda jk/n^{4/3} - jk(j+k)/(2n^2)}(1 + O(1)),$$

so for $\lambda \leq k/n^{2/3}$,

$$\begin{aligned} \sum_{j=k+1}^N \mathbb{E}_{n,\lambda}[Y(k)Y(j)] &\leq \mathbb{E}_{n,\lambda}[Y(k)] \sum_{j=k+1}^N \mathbb{E}_{n,\lambda}[Y(j)]e^{\lambda k^2/n^{4/3} - k^3/n^2}(1 + O(1)) \\ &= \mathbb{E}_{n,\lambda}[Y(k)]\mathbb{E}_{n,\lambda}[Z(k+1)]e^{\lambda k^2/n^{4/3} - k^3/n^2}(1 + O(1)). \end{aligned}$$

Thus (bounding both $\mathbb{E}_{n,\lambda}[Y(k)]$ and $\mathbb{E}_{n,\lambda}[Z(k+1)]$ above by $\mathbb{E}_{n,\lambda}[Z(k)]$)

$$\mathbb{P}(L_1 = k) \geq \mathbb{E}_{n,\lambda}[Y(k)](1 + O(\mathbb{E}_{n,\lambda}[Z(k)]e^{\lambda k^2/n^{4/3} - k^3/n^2})) - \mathbb{P}(L_1 > N).$$

Lemma 19 gives

$$\mathbb{P}_{n,\lambda}(L_1 > N) \leq n^{1/4} \exp(-n^{1/4}/80) = \mathcal{E}_2,$$

and Proposition 13 gives

$$\mathbb{E}_{n,\lambda}[Z(k)] = O\left(\frac{n}{k^{3/2}}e^{-G_\lambda(k^{2/3}/n)}\right).$$

The result follows. \square

Lemma 21. Suppose that $n \geq 81$, $-n^{1/12} \leq \lambda \leq n^{1/12}/5$, and $(3\lambda \wedge 1)n^{2/3} + 1 \leq k \leq n^{3/4}$. Then

$$\mathbb{P}_{n,\lambda}(L_1 \geq k) = \mathbb{E}_{n,\lambda}[Z(k)](1 + O(\mathcal{E}_4)) + O(\mathcal{E}_2).$$

Proof. First note that

$$\mathbb{P}_{n,\lambda}(L_1 \geq k) \leq \mathbb{P}_{n,\lambda}(Z(k) \geq 1) + \mathbb{P}_{n,\lambda}(L_1 > N) \leq \mathbb{E}_{n,\lambda}[Z(k)] + \mathbb{P}_{n,\lambda}(L_1 > N).$$

Lemma 19 tells us that $\mathbb{P}_{n,\lambda}(L_1 > N) \leq \mathcal{E}_2$, so the upper bound is done. For the lower bound, by (9),

$$\mathbb{P}_{n,\lambda}(L_1 \geq k) \geq \mathbb{P}_{n,\lambda}(Z(k) \geq 1) \geq \frac{\mathbb{E}_{n,\lambda}[Z(k)]^2}{\mathbb{E}_{n,\lambda}[Z(k)^2]}.$$

Lemma 18 tells us that

$$\mathbb{E}_{n,\lambda}[Z(k)^2] \leq \mathbb{E}_{n,\lambda}[Z(k)] + \mathbb{E}_{n,\lambda}[Z(k)]^2 \exp\left(\frac{\lambda k^2}{n^{4/3}} - \frac{k^3}{n^2}\right) (1 + O(n^{-1/11})),$$

and Proposition 13 gives

$$\mathbb{E}_{n,\lambda}[Z(k)] = O\left(\frac{n}{k^{3/2}} e^{-G_\lambda(k^{2/3}/n)}\right).$$

Thus

$$\frac{\mathbb{E}_{n,\lambda}[Z(k)]^2}{\mathbb{E}_{n,\lambda}[Z(k)^2]} \geq \mathbb{E}_{n,\lambda}[Z(k)] \left(1 + O\left(\frac{n}{k^{3/2}} e^{-G_\lambda(k^{2/3}/n) + \lambda k^2/n^{4/3} - k^3/n^2}\right)\right) = \mathbb{E}_{n,\lambda}[Z(k)](1 + O(\mathcal{E}_4))$$

as required. \square

Given these lemmas, our first main result follows almost trivially.

Proof of Theorem 1. Lemma 20 tells us that

$$\mathbb{P}_{n,\lambda}(L_1 = k) = \mathbb{E}_{n,\lambda}[Y(k)](1 + O(\mathcal{E}_4)) + O(\mathcal{E}_2).$$

But by Proposition 12,

$$\mathbb{E}_{n,\lambda}[Y(k)] = \frac{k^{1/2}}{(8\pi)^{1/2}n} e^{-G_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_1)).$$

Since $G_\lambda(x) \geq 0$ for all $x \geq 0$, and by the conditions of the theorem we have $\lambda \leq k/n^{2/3}$, we see that $\mathcal{E}_4 \leq n/k^{3/2} \leq \mathcal{E}_1$. This gives part (a) of the result.

For part (b), Lemma 21 gives us

$$\mathbb{P}_{n,\lambda}(L_1 \geq k) = \mathbb{E}_{n,\lambda}[Z(k)](1 + O(\mathcal{E}_4)) + O(\mathcal{E}_2).$$

Then by Proposition 13,

$$\mathbb{E}_{n,\lambda}[Z(k)] = (1 + O(\mathcal{E}_1)) \frac{1}{(8\pi)^{1/2}} \frac{(k/n^{2/3})^{1/2}}{G'_\lambda(k/n^{2/3})} e^{-G_\lambda(k/n^{2/3})} + O(\mathcal{E}_2).$$

Again we observe that $\mathcal{E}_4 \leq \mathcal{E}_1$ which completes the proof. \square

8 The size of the component containing a particular vertex

We now turn to proving Theorem 2, which concerns the component containing a particular vertex v . The proof will be straightforward, relying only on the first moment estimates that we have already developed.

We recall that $\mathcal{C}(v)$ is the connected component containing v , and write $|\mathcal{C}(v)|$ for the number of vertices in $\mathcal{C}(v)$ and $E(\mathcal{C}(v))$ for the number of edges in $\mathcal{C}(v)$. We first state two more results that may be independently useful, before moving on to the proofs of these and Theorem 2.

Proposition 22. *Suppose that $n \geq 16$, $|\lambda| \leq n^{1/12}$, $1 \leq k \leq n^{3/4}$ and $-1 \leq l \leq 4k^{1/2} \wedge 4n^{1/4}$. Then*

$$\begin{aligned} \mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| = k, E(\mathcal{C}(v)) = k + l) &= \frac{\gamma l k^{3l/2}}{(2\pi)^{1/2} n^{l+1}} e^{-F_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_3 + \frac{l^2}{k} + \frac{(l+1)^{1/16}}{k^{9/50}})) \\ &= \left(\frac{3}{8\pi}\right)^{1/2} \frac{1}{n} \left(\frac{ek^3}{12n^2(l \vee 1)}\right)^{l/2} e^{-F_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_3 + \frac{l^2}{k} + \frac{(l+1)^{1/16}}{k^{9/50}} + \frac{1}{l})). \end{aligned}$$

Proposition 23. Suppose that $n \geq 16$ and $|\lambda| \leq n^{1/12}$. For any $M > 0$, there exist constants $0 < c_1 \leq c_2 < \infty$ depending on M such that

$$c_1 k^{-3/2} e^{-F_\lambda(k/n^{2/3})} \leq \mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| = k) \leq c_2 k^{-3/2} e^{-F_\lambda(k/n^{2/3})} \quad \forall k \leq Mn^{2/3}.$$

Everything will be based on the following easy lemma.

Lemma 24. For any $n, k \geq 1$ and $l \geq -1$,

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| = k, E(\mathcal{C}(v)) = k + l) = \frac{k}{n} \mathbb{E}_{n,\lambda}[X(k, k + l)].$$

Proof. We merely note that

$$\begin{aligned} \mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| = k, E(\mathcal{C}(v)) = k + l) &= \binom{n-1}{k-1} C(k, k + l) p_{n,\lambda}^{k+l} (1 - p_{n,\lambda})^{\binom{k}{2} - k - l + (n-k)k} \\ &= \frac{k}{n} \mathbb{E}_{n,\lambda}[X(k, k + l)]. \end{aligned} \quad \square$$

Proof of Proposition 22. Simply combine Lemma 24 with Corollary 8. \square

Proof of Proposition 23. Let $L = \lfloor 4n^{1/4} \rfloor$. By Lemma 24,

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| = k) = \sum_{l=-1}^L \frac{k}{n} \mathbb{E}_{n,\lambda}[X(k, k + l)] + \sum_{l=L+1}^{\infty} \frac{k}{n} \mathbb{E}_{n,\lambda}[X(k, k + l)].$$

By the first part of Proposition 3,

$$\sum_{l=-1}^L \frac{k}{n} \mathbb{E}_{n,\lambda}[X(k, k + l)] = \sum_{l=-1}^L \frac{k}{n} \frac{C(k, k + l)}{(2\pi)^{1/2} n^l k^{k+1/2}} e^{-F_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_3)).$$

Since \mathcal{E}_3 is at most a constant, this means that there exist $0 < c_1 < c_2 < \infty$ such that

$$\frac{c_1 e^{-F_\lambda(k/n^{2/3})}}{n k^{k-1/2}} \sum_{l=-1}^L \frac{C(k, k + l)}{n^l} \leq \sum_{l=-1}^L \frac{k}{n} \mathbb{E}_{n,\lambda}[X(k, k + l)] \leq \frac{c_2 e^{-F_\lambda(k/n^{2/3})}}{n k^{k-1/2}} \sum_{l=-1}^L \frac{C(k, k + l)}{n^l}.$$

Corollary 9 then gives

$$c_1 k^{-3/2} e^{-F_\lambda(k/n^{2/3})} \leq \sum_{l=-1}^L \frac{k}{n} \mathbb{E}_{n,\lambda}[X(k, k + l)] \leq c_2 k^{-3/2} e^{-F_\lambda(k/n^{2/3})}.$$

Thus it remains to show that $\sum_{l=L+1}^{\infty} \frac{k}{n} \mathbb{E}_{n,\lambda}[X(k, k + l)]$ is of smaller order. But this follows easily from the second part of Proposition 3. \square

Proof of Theorem 2. By Lemma 24,

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| = k) = \frac{k}{n} \sum_{l=-1}^{\infty} \mathbb{E}_{n,\lambda}[X(k, k + l)] = \frac{k}{n} \mathbb{E}_{n,\lambda}[Y(k)].$$

Part (a) now follows from Proposition 12.

For part (b) of the theorem, let $N = \lfloor n^{3/4} \rfloor$. By Lemma 24,

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| \geq n^{2/3}, L_1 \leq N) = \sum_{j=k}^N \frac{j}{n} \sum_{l \geq -1} \mathbb{E}_{n,\lambda}[X(j, j + l)] = \sum_{j=k}^N \frac{j}{n} \mathbb{E}_{n,\lambda}[Y(j)].$$

By Proposition 12, this equals

$$\begin{aligned} \sum_{j=k}^N \frac{j^{3/2}}{(8\pi)^{1/2}n^2} e^{-G_\lambda(j/n^{2/3})} (1 + O(\mathcal{E}_1(j, n, \lambda))) \\ = \frac{1}{(8\pi)^{1/2}n} \sum_{j=k}^N \left(\frac{j}{n^{2/3}}\right)^{3/2} e^{-G_\lambda(j/n^{2/3})} (1 + O(\mathcal{E}_1(j, n, \lambda))). \end{aligned}$$

Applying Lemma 16, this is

$$\frac{1}{(8\pi)^{1/2}n^{1/3}} \frac{(k/n^{2/3})^{3/2}}{G'_\lambda(k/n^{2/3})} e^{-G_\lambda(k/n^{2/3})} (1 + O(\mathcal{E}_1(k, n, \lambda))).$$

By Lemma 19, $\mathbb{P}_{n,\lambda}(L_1 > N) = O(\mathcal{E}_2(n))$, so we have shown that

$$\mathbb{P}_{n,\lambda}(|\mathcal{C}(v)| \geq an^{2/3}) = \frac{1}{(8\pi)^{1/2}n^{1/3}} \frac{a^{3/2}}{G'_\lambda(a)} e^{-G_\lambda(a)} (1 + O(\mathcal{E}_1(k, n, \lambda))) + O(\mathcal{E}_2(n))$$

which is (b). □

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